THE GRAM POINTS AND UNIVERSALITY

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Let $\zeta(s)$, $s = \sigma + it$, denote the Riemann zeta-function, and $\theta(t)$, $t > 0$, stand for the increment of the argument of the function $\pi^{-s/2} \Gamma \left( \frac{s}{2} \right)$ along the segment connecting the points $s = \frac{1}{2}$ and $s = \frac{1}{2} + it$. Then the equation $\theta(t) = (n - 1)\pi$, $n \in \mathbb{N}$, for $t > t^* = 6.289 \ldots$ has the unique solution $t_n$. The numbers $t_n$ were introduced by Gram in connection with non-trivial zeros of the function $\zeta(s)$. Now the numbers $t_n$ are called the Gram points. It is known that $t_n \sim \gamma_n$, as $n \to \infty$, where $\gamma_n > 0$ is the imaginary part of non-trivial zeros of $\zeta(s)$.

In [2], Voronin discovered the universality property of the function $\zeta(s)$. Roughly speaking, this means that a wide class of analytic functions defined on the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \}$ can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. If $\tau$ takes values from a discrete set, then the above property is called the discrete universality. Our report is devoted to the discrete universality using the set $\{h t_k : k \in \mathbb{N} \}$, $h > 0$. More precisely, let $\mathcal{K}$ be the class of compact subsets of $D$ with connected complements, and $H_0(\mathcal{K})$, $\mathcal{K} \in \mathcal{K}$, be the class of continuous non-vanishing functions on $\mathcal{K}$ that are analytic in the interior of $\mathcal{K}$.

THEOREM 1. Let $\mathcal{K} \in \mathcal{K}$ and $f(s) \in H_0(\mathcal{K})$. Then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{s \in \mathcal{K}} |\zeta(s + iht_k) - f(s)| < \varepsilon \right\} > 0.$$ 

References

1. J.-P. Gram, Sur les zéros de la fonction $\zeta(s)$ de Riemann, Acta Math. 27 (1903), 289–304. MR1554986

2. S.M. Voronin, Theorem on the “universality” of the Riemann zeta-function, Math. USSR Izv. 9(3) (1975) 443–453. MR0941684

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