There are several ways that random trees can be generated by branching processes, and this is also an important tool to study several types of random trees that are defined in different ways. I focus on two such connections, with conditioned Galton–Watson processes and with general CMJ branching processes; both versions yield many classes of random trees studied in other contexts. These ideas are old, and go back at least to Devroye [3–5], and Aldous [1, 2]; I will present some old and some new results, mainly from [6, 7].

**Simply generated trees and conditioned Galton–Watson processes**

I will define a *simply generated random tree* and a *conditioned Galton–Watson tree*. Every conditioned Galton–Watson tree is a simply generated random tree. Conversely, many important examples of simply generated random trees are conditioned Galton–Watson trees, but there are also some exceptional simply generated random trees. To be precise, there are three cases for a simply generated random tree: it is either (i) equivalent to a critical conditioned Galton–Watson tree, (ii) equivalent to a subcritical conditioned Galton–Watson tree but not to a critical one, (iii) not equivalent to any conditioned Galton–Watson tree.

In all three cases, the tree converges in distribution as the size $n \to \infty$ in a local topology (at the root), to a random infinite tree, but this limit tree is of different types in the three cases.

Other limit theorems concern the global structure, after suitable rescaling.

A third family of results concern the local structure far away from the root. Such results can often be formulated in terms of random *fringe trees*; a fringe tree $T_v$ is the subtree consisting of a node $v$ and all its descendants. The random fringe tree of a conditioned Galton–Watson tree converges in distribution to the corresponding (unconditioned) Galton–Watson tree. Furthermore, assuming that the offspring distribution has a finite variance, the fluctuations are asymptotically normal; thus, for example, the number of fringe trees isomorphic to a given tree is asymptotically normal. Several important statistics (functionals) of trees can be written as a sum over all fringe trees of another functional. This leads, under suitable conditions, to central limit theorems.

The non-fringe subtrees of a conditioned Galton–Watson tree behave quite differently than the fringe subtrees. The number of non-fringe subtrees is exponentially large, with an asymptotic log-normal distribution, and a typical non-fringe subtree is large.

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Family trees of general CMJ branching processes

Another class of random trees with a given size is obtained by taking the family tree of a (supercritical) Crump–Mode–Jagers process, and stopping at the time $t$ that the tree reaches the desired size. This (typically) yields trees where the height is logarithmic in the size $n$, while the conditioned Galton–Watson trees (typically) have height of the order $\sqrt{n}$. Asymptotic results for the height of the tree and depth of individual nodes follow by branching process theory.

A random fringe tree converges in distribution also in this situation, with a limit that is given by the CMJ process stopped at an independent exponential random time. This can be extended to the extended fringe tree, where we consider also the parent, siblings, grandparent, and so on of a random node.

However, so far no corresponding central limits theorems are known. In fact, we conjecture that functionals such as the number of fringe trees of a given type are asymptotically normal in some cases but not in all. This leads to similar problems about fluctuations for functionals of CMJ branching processes; these problems are largely still open, although we have some preliminary results.

References