A SHARP BOUND ON THE EXPECTED NUMBER OF UPCODESINGS OF AN $L_2$-BOUNDED MARTINGALE

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Consider a martingale $(M_n)_{n \geq 0}$, starting at $x$. It is well known that for any martingale it holds $E[M_n] = E[M_0] = x$. Now, let $M$ be $L_2$-bounded and assume that the final coordinate or limit $Y$ be such that $\text{Var}(Y) = \sigma^2$. We investigate how much variability $M$ can have, allowed by $\sigma^2$. In the past, Dubins and Schwarz [1] proved that $E[\max(M)] = \sigma$ and $E[\max|M|] = \sigma \sqrt{2}$, while Dubins, Gilat and Meilijson [2] showed that $E[\max(M) - \min(m)] = \sigma \sqrt{3}$.

Here, we give another bound on the variability of the considered martingale in terms of the expected number of up-crossings of an interval. In particular, we prove that the upper bound for the expected number of up-crossings of $(a, b)$ by $M$ is $\sigma/2$ and that this bound is attained by a martingale starting at $x = a$. To prove this result, we use the Doob’s up-crossing inequality.

In our approach we denote with $\Delta = (b - a)/\sigma$ and $\delta = |x - a|/\sigma$ the normalized length of the interval and of the distance from the initial point to the lower endpoint, respectively. Then we prove that the expected number of up-crossings of $(a, b)$ by $M$ is at most $\sqrt{1 + \delta^2}/2\Delta$ if $\Delta^2 \leq 1 + \delta^2$, and at most $1/(1 - (\Delta + \delta)^2)$ otherwise. Both bounds are sharp, attained by a Standard Brownian Motion stopped at appropriate times. Furthermore, we show that both bounds attain the Doob’s upper bound on the expected number of up-crossings of $(a, b)$.

References