ON BACKWARD KOLMOGOROV EQUATION RELATED TO CIR PROCESS

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We consider the existence of a classical smooth solution of the backward Kolmogorov equation
\[
\begin{cases}
\partial_t u(t,x) = A u(t,x), & x \geq 0, t \in [0,T], \\
u(0,x) = f(x), & x \geq 0,
\end{cases}
\]
where $A$ is the generator of the CIR process, the solution of the stochastic differential equation
\begin{equation}
X^t_x = x + \int_0^t \theta (\kappa - X^s_x) \, ds + \sigma \int_0^t \sqrt{X^s_x} \, dB_s, \quad x \geq 0, \quad t \in [0,T],
\end{equation}
that is, $Af(x) = \theta (\kappa - x) f'(x) + \frac{1}{2} \sigma^2 x f''(x), \quad x \geq 0$ ($\theta, \kappa, \sigma > 0$). Alfonsi [1] showed that this equation has a smooth solution with partial derivatives of polynomial growth, provided that the initial function $f$ is smooth with derivatives of polynomial growth. This fact is essential in rigorous proofs of the convergence rates of weak approximations of Eq. (1) (and of general SDEs as well). The proof of Alfonsi is mainly based on the analytical formula for the transition density of the CIR process in the form of a rather complicated function series. We present a direct proof for a CIR process satisfying the condition $\sigma^2 \leq 4 \theta \kappa$. It is based on the representation of a CIR process in terms of a squared Bessel process and its additivity property. We believe that our approach will be applicable to a wider class of “square-root-type” processes, for which an explicit form of the transition function is not known (e.g., the well-known square-root stochastic-volatility Heston process).

References